Study of field fluctuations and their localization in a thick braneworld generated by gravity non–minimally coupled to a scalar field with a Gauss-Bonnet term

Gabriel Germán,[a] **Alfredo Herrera–Aguilar,**[a,b] **Dagoberto Malagón–Morejón,**[a,b] **Israel Quiros**[c] **and Roldão da Rocha**[d]

[a]Instituto de Ciencias Físicas, Universidad Nacional Autónoma de México.
  Apdo. Postal 48–3, 62251, Cuernavaca, Morelos, México.

[b]Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo.
  Edificio C–3, Ciudad Universitaria, C.P. 58040, Morelia, Michoacán, México.

[c]Departamento de Matemáticas, Centro Universitario de Ciencias Exactas e Ingenierías,
  Corregidora 500 S.R., Universidad de Guadalajara, 44420 Guadalajara, Jalisco, México.

[d]Centro de Matemática, Computação e Cognição, Universidade Federal do ABC.
  Rua Santa Adélia, 166 09210-170, Santo André, SP, Brazil.

  E-mail: gabriel@fis.unam.mx, aha@fis.unam.mx, malagon@ifm.umich.mx, iquiros@fisica.ugto.mx, roldao.rocha@ufabc.edu.br

Abstract. In this work we study a scenario with a warped 5D smooth braneworld with 4D Minkowski geometry builded from bulk scalar matter non–minimally coupled to gravity with an additional Gauss–Bonnet term. We present exact solutions for the full braneworld configuration in contrast to previous results where only approximate solutions were constructed due to the highly non–linear character of the relevant differential equations. These solutions allow us to study the necessary conditions for the finiteness of the 4D Planck mass and the 4D effective cosmological constant. This fact enables us to perform a more rigorous analysis of 4D gravity localization compared to approximate approaches. We also analyze the localization properties of scalar, vector and tensor fluctuation modes for the constructed field configurations. We show that for the considered backgrounds, only the massless tensor mode, i.e. the 4D graviton, is localized on the brane, while the vector and scalar modes are not confined to the brane.

Keywords: Thick branes, Gauss–Bonnet term, non–minimal coupling, gravity localization, metric fluctuations.

Contents

1 Introduction

Braneworlds models [1–9] are an interesting alternative to the standard Kaluza–Klein compactification. In order to protect the standard 4D physics of some unobserved effects, in the Kaluza–Klein paradigm extra dimensions are compactified to a tiny size (for a 4D observer that is equivalent to the very hight energy regime). The braneworld scenario has another approach, in this alternative the physics of our 4D world is compatible with the existence of infinite extra dimensions. A procedure to do that is by considering that the standard model (SM) fields are trapped on a 4D hypersurface, called a 3-brane (our universe). In contrast with the ordinary matter, gravity and exotic matter can reside in the whole higher-dimensional manifold (bulk). Although the gravitational field can propagate through all dimensions, one of the first requirements to have realistic braneworlds models is to recover standard 4D gravity

on the brane. For example, if the induced geometry on the brane is flat we need to localized a 4D graviton on our brane.

The braneworlds models is classified with respect to their width in thin and thick branes. In a scenario with a thin braneworld [7–9] the curvature scalars are singular at the location of the branes, that is direct consequence of the null width of the branes. In spite of that complication, the called Israel-Lanczos junction conditions [10–12] make harmless such singularities when we study the localization of gravity and matter and in calculation of several effective parameters of the model i.e. 4D Planck mass, etc.

The fact of considering that the ordinary matter fields are completely confined on a region with null width is just an approximation. The phenomenology of the Standard Model are well known at electro–weak energy scale m_{EW}, therefore, although the standard matter can not move freely through the extra dimensions, it is possible that they can access to distant $r \leq m_{EW}^{-1} \sim 10^{-19}$ m without contradictions with the Standard Model [5]. In other words, there are more realistic alternatives to thin brane configurations they are known, generically, as thick braneworlds or domain walls. When the brane has a non–trivial width it is not necessary to use the Israel-Lanczos junction conditions, and this might be a mathematical advantage whether the action of the model is complicated (too many matter fields, non–minimal coupling, higher order curvature terms, etc).

Thick branes might be constructed by using different procedures. One of the more evident example is by replacing the delta functions in the action by a non–singular source functions. Another simple method is by using self-interacting scalar fields coupled minimally to gravity [13–21].

Within the frame of thin (Randall-Sundrum) braneworlds, bulk scalar fields have been investigated, for instance, in [22–26], where the Randall-Sundrum scenario has been modified by considering self-interacting scalar fields non–minimally coupled to gravity [22]. Furthermore, in [23] and [24] the resulting action has been modified by adding a Gauss-Bonnet term. The influence of higher curvature terms in scalar-field-generated thick brane models has been studied in Refs. [27–37].

In dimensions higher than four (in 5D, for instance) the standard Einstein-Hilbert action may be supplemented with higher order curvature corrections without generating, in the equations of motion, terms containing three or higher order derivatives of the metric with respect to the space-time coordinates [38, 39]. A particular combination which satisfied that requirement is the well-known Gauss-Bonnet (GB) invariant:

$$\mathcal{R}_{\mathrm{GB}}^2 = R^{ABCD}R_{ABCD} - 4R^{AB}R_{AB} + R^2, \tag{1.1}$$

where $A, B, C, D = 0, 1, 2, 3, 5$. While in 4D the GB invariant is a topological term which does not contribute to the classical equations of motion[1], in dimensions higher than four the GB combination leads to a ghost-free theory and it appears in different higher dimensional contexts. For instance, in string theory the GB-term appears in the first string tension correction to the (tree-level) effective action [41–47]. In the framework of thick braneworld scenarios generated by a self-interacting minimally coupled scalar field, the GB invariant has been studied in connection with the localization properties of the various modes of the geometry in Ref. [27] (see also, for instance, [36]). The non-minimal interaction between a bulk (self-interacting) scalar field and gravity has been considered, for instance, in Ref. [48], where the perturbative stability of a thick brane was explored.

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[1]In scenarios with AdS_4 the 4D Gauss–Bonnet term produces non–trivial results in the conserved quantities of the theory [40].

It seems quite natural to investigate the effect of a self-interacting scalar field non–minimally coupled to gravity [49–51], in thick braneworld scenarios with higher curvature terms (a Gauss-Bonnet invariant, for instance). In this work we study a 5D thick braneworld modeled by a smooth scalar domain wall non–minimally coupled to gravity with a Gauss-Bonnet term on the bulk. Our results are a extension and generalization of previous results obtained within the framework of braneworlds with 4D Poincaré symmetry generated by scalar fields minimally coupled to gravity [52] (by using a similar approach, thick braneworlds with de Sitter 3-branes were studied in [53]). In order to accomplish that, we first construct exact solutions for the highly non–linear differential equations for the scalar field, a necessary step for providing a rigorous study of the consistency and localization properties of the metric fluctuations of our braneworld configuration. Moreover, these solutions are essential for analyzing the positiveness of the 4D Planck mass (an indispensable property of any viable theory). We further analyze the tensor perturbations of the geometry which can be classified into scalar vector and tensor sectors according to the 4D Poincaré symmetry group. We establish that both the scalar and vectorial sectors are not localized on the brane in contrast to the 4D massless mode, the 4D graviton, of the tensorial fluctuation modes. We finally summarize our results and conclusions at the end of the paper.

2 The model

Let us explore a thick braneworld described by the following 5D action (similar set up is studied in references [27], [48]):

$$S = - \int d^5 x \sqrt{|g|} \left[\frac{L}{2\kappa} R - \frac{1}{2} (\nabla \varphi)^2 + V + \alpha' \mathcal{R}_{\mathrm{GB}}^2 \right], \qquad (2.1)$$

where $\alpha' > 0$, and $\kappa \simeq 1/M^3$, where M – the 5D Planck mass. Besides, φ is a real scalar field and $V = V(\varphi)$ its self–interaction potential. The quantity $L = L(\varphi)$ describes the non–minimal coupling between the scalar field φ and the Einstein–Hilbert term and $\mathcal{R}_{\mathrm{GB}}^2$ is the 5D Gauss-Bonnet term (1.1).

In the scalar-tensor gravity theory (2.1), 15 degrees of freedom g_{AB}, plus the scalar field φ, propagate the gravitational interaction. Hence, this is not a pure geometrical theory of gravity. In particular, the metric coefficients define the geodesic motion of test particles, while the scalar field determines locally the strength of gravity by means of the effective gravitational coupling $\propto \kappa/L(\varphi)$.

The way the scalar field φ is coupled to the curvature in the present theory, deserves an independent comment. Actually, in this paper, just as a matter of necessary simplicity, we have considered explicit coupling of φ to the curvature scalar R, but not to the Gauss-Bonnet invariant. Intuition, instead, dictates that the scalar field should couple in a similar fashion to R and to $\mathcal{R}_{\mathrm{GB}}^2$, since both contribute towards the curvature of space-time. This will be the subject of forthcoming work. We want to underline that this 'asymmetric' coupling has nothing to do with physical requirements, but just with simplicity of further mathematical handling since, as we shall see further, the relevant equations of motion are highly non–linear.

The Einstein's field equations that come from the action (2.1) take the following form:

$$L \, R_{AB} + \epsilon \, \mathcal{Q}_{AB} = \kappa \, \tau_{AB} + \nabla_A \nabla_B L + \frac{1}{3} \, g_{AB} \, \square L, \qquad (2.2)$$

where,

$$\mathcal{Q}_{AB} = \frac{1}{3}\, g_{AB}\, \mathcal{R}_{\text{GB}}^2 - 2\, R\, R_{AB} + 4\, R_{AC}\, R^C_B + 4 R^{CD}\, R_{ACBD} - 2\, R_{ACDE}\, R_B^{\ CDE},$$

is the Lanczos tensor representing the Gauss-Bonnet corrections to the Einstein's field equations, $\epsilon = 2\alpha'\kappa$, $\square \equiv g^{CD}\nabla_C\nabla_D$, and the stress-energy tensor τ_{AB}, corresponding to the scalar matter content on the bulk, is defined as usual:

$$\tau_{AB} = \partial_A\varphi\partial_B\varphi - \frac{2}{3}\, g_{AB}\, V(\varphi).$$

The remaining terms in the right-hand-side (RHS) of (2.2) come from the non–minimal coupling of the scalar field to the curvature scalar.

Furthermore, the equation that describes the dynamics of the scalar field (Klein-Gordon equation) can be written as follow:

$$\square\varphi + \frac{1}{2\kappa}\, R\, L_\varphi + \frac{\partial V}{\partial\varphi} = 0, \tag{2.3}$$

where, $L_\varphi = dL/d\varphi$. As it is done, for instance, in Ref. [54], the geometry of our braneworld model is described by a warped metric in conformally flat coordinates:

$$ds^2 = a^2(w)[\eta_{\mu\nu}dx^\mu dx^\nu - dw^2], \tag{2.4}$$

where the variable w is the extra coordinate, and $\eta_{\mu\nu}$ is the 4D Minkowski metric. Due to the simplicity of our metric ansatz, it is natural to consider that the field φ depends only on the fifth coordinate w. By taken into account the last comment and (2.4) the field equations (2.2) and (2.3) read

$$V + \frac{3\,\mathcal{H}\, L_\varphi\, \varphi'}{\kappa\, a^2} + \frac{1}{2\kappa\, a^2}\left(\varphi''\, L_\varphi + \varphi'^2\, L_{\varphi\varphi}\right)$$
$$-\frac{3}{2\kappa a^2}\left[\frac{4\,\epsilon}{a^2}\,\mathcal{H}^2\,\left(\mathcal{H}^2 + \mathcal{H}'\right) - \left(\mathcal{H}' + 3\mathcal{H}^2\right)\, L\right] = 0, \tag{2.5}$$

$$\frac{1}{\kappa}L_\varphi\, \varphi'' - \frac{2}{\kappa}\,\mathcal{H}\, L_\varphi\, \varphi' + \left(1 + \frac{L_{\varphi\varphi}}{\kappa}\right)\varphi'^2 - \frac{3}{\kappa}\left(\mathcal{H}^2 - \mathcal{H}'\right) q = 0,$$

$$\varphi'' + 3\mathcal{H}\varphi' - \frac{dV}{d\varphi}a^2 - \frac{2L_\varphi}{\kappa}\left(3\mathcal{H}^2 + 2\mathcal{H}'\right) = 0.$$

Here the tilde denotes derivative with respect to the extra-coordinate, in addition to this

$$\mathcal{H} \equiv \frac{a'}{a}, \qquad \text{while} \qquad q \equiv L - \frac{4\,\epsilon}{a^2}\,\mathcal{H}^2.$$

As one can straightforwardly check the equations (2.5) are not independent.

3 Planck masses

The 4D effective theory can be obtained by integrating the 5D action (2.1) with respect to the fifth coordinate w (dimensional reduction). In order to do that let us consider a generalization of the 5D line-element (2.4)

$$ds^2 = a^2(w)[\tilde{g}_{\mu\nu}(x)dx^\mu dx^\nu - dw^2], \tag{3.1}$$

where, $\tilde{g}_{\mu\nu}(x)$, is an arbitrary 4D metric. The 5D fundamental theory is reduced to a 4D Einstein-Hilbert effective action plus the corrections that come from the scalar matter and higher curvature terms of the bulk. In this section we will focus in the analysis of the 4D Planck mass, therefore we only need to extract the 4D Einstein-Hilbert effective after the dimensional reduction of the 5D action[2]

$$S_4 \simeq M_{\rm Pl}^2 \int d^4x \sqrt{|\tilde{g}_4|}\tilde{R}_4 + \cdots , \tag{3.2}$$

where the subscript 4 labels quantities computed with respect to 4D metric, $\tilde{g}_{\mu\nu}(x)$ and $M_{\rm Pl}$ is the effective Planck mass.

A method to find the low dimensional effective theory and in particular the 4D Einstein-Hilbert action consists in interpreting the warp factor as a conformal function as follows [55]:

$$\tilde{g}_{AB} \mapsto g_{AB} = a^2(w)\tilde{g}_{AB} = a^2(w) \begin{pmatrix} \tilde{g}_{\mu\nu}(x) & 0 \\ 0 & -1 \end{pmatrix},$$

and, rewriting the 5D action in terms of the quantities defined with respect to $\tilde{g}_{AB}$. In order to obtain the Einstein-Hilbert part of the 4D effective action, it is only necessary to consider the above conformal transformation on the terms, $-L(\varphi)\,R/2\kappa$, and, $\alpha'\mathcal{R}_{\rm GB}^2$, in (2.1). The following expressions display these quantities after the conformal transformation,

$$R = a^{-2}\left(\tilde{R} - 8\tilde{\Box}\vartheta - 12(\tilde{\nabla}\vartheta)^2\right), \tag{3.3}$$

$$\begin{aligned}
\mathcal{R}_{GB}^2 = \frac{1}{3a^4}\Big\{ &\tilde{\mathcal{R}}_{GB}^2 - 12\tilde{R}(\tilde{\nabla}\vartheta)^2 - 24\tilde{R}\tilde{\Box}\vartheta + 48\tilde{R}^{AB}\left(\tilde{\nabla}_A\tilde{\nabla}_B\vartheta - \tilde{\nabla}_A\vartheta\tilde{\nabla}_B\vartheta\right) \\
&+ 72\left[(\tilde{\Box}\vartheta)^2 + (\tilde{\nabla}\vartheta)^4 - (\tilde{\nabla}_A\tilde{\nabla}_B\vartheta)(\tilde{\nabla}^A\tilde{\nabla}^B\vartheta)\right], \\
&+ 144(\tilde{\nabla}_A\tilde{\nabla}_B\vartheta)(\tilde{\nabla}^A\vartheta)(\tilde{\nabla}^B\vartheta) + 144(\tilde{\nabla}\vartheta)^2\tilde{\Box}\vartheta\Big\},
\end{aligned} \tag{3.4}$$

where, $\vartheta = \ln a$. The terms with a tilde are defined with respect to the metric $\tilde{g}_{AB}$.

The relation between 4D and 5D Planck masses can be written as follow

$$\begin{aligned}
M_{\rm Pl}^2 &\simeq M^3 \int_{-\infty}^{\infty} a^3(w)\left[L + \frac{4\epsilon}{a^2}(\mathcal{H}^2 + 2\mathcal{H}')\right] dw, \\
&= M^3 \int_{-\infty}^{\infty} a^3(w)\, q\, dw + 8\,M^3\,\epsilon\,[a']_{-\infty}^{\infty}.
\end{aligned} \tag{3.5}$$

This quantity should be positive and finite for a consistent theory. Moreover, it should reproduce the effective gravitational couplings that we observe in our 4D world, if we wish to recover 4D gravity on the brane. Later on we shall see that these conditions are fulfilled for a wide class of solutions of our scalar-tensor model.

4 Background Geometry and Non-minimal Coupling

Here we shall consider the following coupling function [22] (see also [48]):

$$L(\varphi) = 1 - \frac{\xi}{2}\varphi^2. \tag{4.1}$$

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[2]Later on we will study the whole 4D effective theory that arises after integrating the 5D action with respect to the extra–dimensional coordinate.

If the parameter $\xi = 0$, the scenario is described by a bulk scalar field minimally coupled to gravity (it was explored in Ref. [27]). Hence, the parameter ξ characterize models with non–minimal coupling between scalar matter and gravity. Positivity of L leads to φ^2 being bounded from above:

$$L \geq 0 \quad \Rightarrow \quad \varphi^2 \leq \frac{2}{\xi} \quad \Rightarrow \quad |\varphi| \leq \sqrt{\frac{2}{\xi}}. \tag{4.2}$$

We shall consider, besides, a regular geometry of the form (2.4), which interpolates between two asymptotically AdS_5 space-times, which is depicted by the following warp factor [27]:

$$a(w) = \frac{a_0}{\sqrt{1 + b^2 w^2}}, \tag{4.3}$$

where the width of the thick brane is $1/b$, and the quantity a_0 is dimensionless and related to the radius of the asymptotic AdS_5 space-time. All of the resulting quadratic curvature invariants are regular and asymptotically constant.

Let us make a change of variable $w \to x = bw$ where x is dimensionless. By using (4.1) the second equation in (2.5) can be rewritten as:

$$\xi\varphi\varphi'' + \frac{2\xi\,x}{1+x^2}\,\varphi\varphi' + (\xi - \kappa)\varphi'^2 - \frac{3\xi}{2}\frac{\varphi^2}{(1+x^2)^2}$$
$$= -\frac{3}{(1+x^2)^2} + \frac{4\epsilon b^2}{a_0^2}\frac{3x^2}{(1+x^2)^3}, \tag{4.4}$$

where, now, the tilde denotes derivative with respect to the dimensionless variable x. It is difficult to find the general solution to the above equation, however, several interesting (particular) exact solutions can be found. Here we shall split the analysis into three separated cases corresponding to: i) a scalar field minimally coupled to gravity and including a 5D Gauss-Bonnet invariant ($\xi = 0$ and $\epsilon \neq 0$) [27], ii) a scalar field non–minimally coupled to curvature without the Gauss-Bonnet term ($\xi \neq 0$ and $\epsilon = 0$), and iii) the more general situation when both non–linear effects are present, i.e. when $\xi \neq 0$ and $\epsilon \neq 0$. Once a given particular (exact) solution for the scalar field $\varphi = \varphi(x)$ is found, one hence can write the self-interaction potential as a function of the (dimensionless) extra-dimensional coordinate x:

$$V(x) = \frac{3b^2}{2\kappa a_0^2}\left\{\frac{4\epsilon b^2}{a_0}\frac{x^2(2x^2-1)}{(1+x^2)^2} + \frac{1-4x^2}{1+x^2} + \xi\left[(1+x^2)(\varphi\varphi'' + \varphi'^2) - 2x\varphi\varphi'\right.\right.$$
$$\left.\left. - \left(\frac{1-4x^2}{1+x^2}\right)\frac{\varphi^2}{2}\right]\right\}. \tag{4.5}$$

The above is just a rewriting of the first equation in (2.5) for the case of interest in this paper.

Once φ, and V, are given as functions of the dimensionless variable x, the components of the stress energy tensor $\tau_{AB} = \tau_{AB}(x)$, are also known functions of the dimensionless extra-dimensional variable.

5   Mirror Symmetry

An important remark on the particular solutions we are looking for is related to the symmetries of the metric coefficients. As mentioned, in the general case when there is a non–minimal coupling between the scalar field and the curvature scalar ($\xi \neq 0 \Rightarrow L = L(\varphi) \neq 1$),

gravity is propagated both by g_{AB} and by φ. Hence, one should naively expect that the (real) scalar field, $\varphi(x)$, will respect the same symmetries as the metric functions g_{AB}. In the remaining part of this section we shall show that, as a matter of fact, while the latter is a mandatory requirement for the case when $\xi = 0$ (in the minimal coupling case), this property is not present in the general case in which $\xi \neq 0$.

In the present case the metric coefficients in (2.4) inherit the 'mirror' symmetry

$$x \mapsto -x \quad \Rightarrow \quad g_{AB}(x) = g_{AB}(-x), \tag{5.1}$$

distinctive of the warp factor (4.3). In consequence, the tensors R_{AB}, and $\mathcal{Q}_{AB}$, in the left-hand-side (LHS) of Einstein's field equations (2.2)

$$R_{AB}(x) = R_{AB}(-x), \qquad \mathcal{Q}_{AB}(x) = \mathcal{Q}_{AB}(-x), \tag{5.2}$$

also respect invariance under (5.1).

For the minimal coupling case ($\xi = 0 \Rightarrow L = 1$), since the Einstein's field equations read,

$$R_{AB} + \epsilon \mathcal{Q}_{AB} = \kappa \tau_{AB}, \tag{5.3}$$

the invariance (5.2) will entail that, necessarily, $\tau_{AB}(x) = \tau_{AB}(-x)$, which means, in turn, that

$$V(x) = V(-x), \qquad \varphi(x) = \pm\varphi(-x), \tag{5.4}$$

where, in the last equality, only one of the two signs, "+", or "−", is to be chosen at once. In other words, φ can be either even or odd, under (5.1).

In the general case $\xi \neq 0$ ($L = L(\varphi) \neq 1$), on the contrary, since the coupling function L is multiplying the Ricci tensor in the LHS of (2.2), mirror symmetry (5.1), (5.2) is respected only if the additional requirement

$$L(\varphi(x)) = L(\varphi(-x)), \tag{5.5}$$

is fulfilled.[3] In the case studied in this paper, since L is quadratic in φ (see Eq. (4.1)), the latter requirement will entail, again, that, under (5.1), φ can be either even, or odd.

In case φ does not show any obvious symmetry under (5.1),

$$\varphi(x) \neq \pm\varphi(-x) \quad \Rightarrow \quad L(\varphi(x)) \neq L(\varphi(-x)),$$

the coupling function L counteracts the symmetry displayed by R_{AB} and, in consequence, the LHS of (2.2) – the pure geometrical part of Einstein's equations – is not invariant under (5.1) anymore. Hence, if $\xi \neq 0$, mathematical consistency of the solutions does not impose any mirror symmetry requirement on φ. This can be only an independent 'ad hoc' requirement.

Notwithstanding, in the present paper the symmetry requirements (5.1), (5.2), (5.5), are be used to select physically suitable solutions among the set of all possible exact solutions we will be able to find. Consequently, we shall take into consideration only those φ-configurations which are either even ($\varphi(x) = \varphi(-x)$) or odd ($\varphi(x) = -\varphi(-x)$) under $x \mapsto -x$. While for the minimally coupled case it is legitimate, as shown above, in the case when $\xi \neq 0$, it is not required by mathematical consistency, but, by symmetric aesthetic, instead.

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[3]In our analysis we took into account the fact that the operators $\nabla_A \nabla_B$, and $\Box$ in the RHS of Eq. (2.2), are invariant under (5.1). Recall that, in the present case, there is functional dependence on the extra-coordinate x only.

6 Four-dimensional effective theory and cosmological constant

In section 3 we obtained the 4D Planck mass in terms of the 5D parameters. As mentioned above, after one performs a dimensional reduction of (2.1) the 5D theory is reduced to a 4D Einstein-Hilbert effective action plus the corrections that come from the scalar matter and higher curvature terms of the bulk. Thus, the 4D effective theory must be well defined. It is not enough to have a finite and positive 4D Planck mass, e.g., the 4D cosmological constant has to be finite as well, otherwise the model (2.1) will not reproduce the physics of our world. Let us study the whole effective theory that arises after the dimensional reduction of the fundamental model (2.1). By considering the metric (3.1), the Ricci and Gauss–Bonnet term (3.4) take the form

$$R = \frac{1}{a^2}(\tilde{R} + 8\mathcal{H}' + 12\mathcal{H}^2), \tag{6.1}$$

$$\mathcal{R}^2_{GB} = \frac{1}{3a^4}(\tilde{\mathcal{R}}^2_{GB} + 72\mathcal{H}^2(\mathcal{H}^2 + 4\mathcal{H}')) + \cdots, \tag{6.2}$$

where the $\cdots$ represent terms associated to the 4D Einstein–Hilbert action that were studied in section 3.

Furthermore, $\tilde{R}_{5ABC} = \tilde{R}_{5A} = 0$, thus, the non–trivial components of the Riemann and Ricci tensor are the associated to the 4D metric $\tilde{g}_{\mu\nu}(x)$. In order to simplify the analysis let us assume that the metric $\tilde{g}_{\mu\nu}(x)$ is a small deviation from $\eta_{\mu\nu}$, in other words, $\tilde{g}_{\mu\nu}(x) = \eta_{\mu\nu} + \delta g_{\mu\nu}(x)$, while we can set $\varphi \to \tilde{\varphi} = \varphi + \delta\varphi_4(x)$, where φ is the fundamental scalar field and $\delta\varphi_4(x)$ is its 4D part. The effective action in four dimensions can be written as follows

$$S_{eff} = S = -\int d^4x \sqrt{|\tilde{g}_4|}\left(\Lambda_4 - \frac{\lambda_1}{2}(\tilde{\nabla}_4\varphi_4)^2 + \lambda_2 U_4(\varphi_4) + \lambda_3\tilde{\mathcal{R}}^2_{4GB}\right) + S_4, \tag{6.3}$$

where the subindex 4 is associated to the metric $\tilde{g}_{\mu\nu}(x)$, S_4 is 4D Einstein–Hilbert action (3.2) and $U_4(\varphi_4)$ is the self–interaction potential associated to the field $\varphi_4 = \delta\varphi_4(x)$. On the other hand, Λ_4, λ_1, λ_2 and λ_3 are effective coupling constants and take the following form

$$\Lambda_4 = \frac{1}{\kappa}\int dw \left(2a^3 L(2\mathcal{H}' + 3\mathcal{H}^2) + \frac{\kappa a^3 \varphi'^2}{2} + \kappa a^5 V + 12\epsilon a\mathcal{H}^2(4\mathcal{H}' + \mathcal{H}^2)\right),$$

$$\lambda_1 = \int a^3 dw,$$

$$\lambda_2 = \int a^5 dw,$$

$$\lambda_3 = \frac{\alpha'}{3}\int a\, dw.$$

The low dimensional theory (6.3) is described by a 4D scalar field minimally coupled to gravity plus the Gauss–Bonnet term in four dimensions. Furthermore, Λ_4 is the 4D cosmological constant. It is not difficult to show that the matter coupling λ_1 and λ_2 are finite for the metric (4.3). Similarly to the Planck mass (3.5), the effective cosmological constant Λ_4 depends on the fundamental scalar field φ. In the case where $|\varphi|$, $|\varphi'|$ and $|V|$ are bounded along the extra dimension (these requirements are satisfied for the solutions that we will study in this work) the effective cosmological constant is finite. Otherwise, λ_3 is unbounded. Therefore, the effects of the 4D Gauss–Bonnet term are decoupled of the 4D phenomenology. Maybe this is not as harmful as it seems, if one takes into account the fact that the contribution of the Gauss–Bonnet term to dynamical equations is trivial in four dimensions for almost all the spacetimes.

7 Exact solutions

As already mentioned, here we shall split the study into three particular cases i) $\epsilon \neq 0$ and $\xi = 0$ – minimal coupling, ii) $\xi \neq 0$ and $\epsilon = 0$ – non-minimal coupling, and iii) $\epsilon \neq 0$ and $\xi \neq 0$ – the general case. We will rely on a number of assumptions, so that, after achieving considerable simplification, particular exact solutions can be found.

7.1 Minimal Coupling Case ($\epsilon \neq 0$ and $\xi = 0$)

If one sets, $\xi = 0$, then (4.4) can be easily solved. Actually, let us first change to the variable

$$\tau = \arctan(x) \quad \Leftrightarrow \quad x = \tan(\tau), \qquad \sin^2 \tau = \frac{x^2}{1 + x^2},$$

$$(1 + x^2)\frac{d}{dx} = \frac{d}{d\tau} \quad \Leftrightarrow \quad (1 + x^2)X' = \dot{X}. \tag{7.1}$$

Notice that $x \in \,]-\infty, +\infty[\, \Rightarrow \tau \in \,]-\pi/2, \pi/2[$. In terms of this new variable equation (4.4), with $\xi = 0$, it transforms into the following first order equations:

$$\dot{\varphi} = \pm\sqrt{\frac{3}{\kappa}}\sqrt{1 - \frac{4\epsilon b^2}{a_0^2}\sin^2 \tau}, \tag{7.2}$$

which can be integrated in quadratures to give:

$$\varphi^{\pm}(\tau) = \pm\sqrt{\frac{3}{\kappa}}\, E(\sin\tau, k) + \varphi_0^{\pm}, \tag{7.3}$$

where

$$E(\sin\tau, k) = \int_0^\tau \sqrt{1 - k^2\sin^2\zeta}\, d\zeta, \qquad k = \sqrt{\frac{4\epsilon b^2}{a_0^2}},$$

is the incomplete elliptic integral of the second kind [56], $\varphi_0^{\pm}$ are arbitrary integration constants, and the $\pm$ signs represent different branches of the solution.

The particular case when, $a_0 = \sqrt{4\epsilon}b \Rightarrow k = 1$ [54], $\varphi^{\pm}(\tau) = \pm\sqrt{3/\kappa}\,\sin\tau + \varphi_0^{\pm}$, or, in terms of the dimensionless variable x (see Eq. (7.1)):

$$\varphi^{\pm}(x) = \pm\sqrt{\frac{3}{\kappa}}\frac{x}{\sqrt{1 + x^2}} + \varphi_0^{\pm}. \tag{7.4}$$

As before, the $\pm$ signs describe two possible branches of the solution. To fix the constants $\varphi_0^{\pm}$, one has to choose appropriate boundary conditions. One simple way is by choosing, for instance,

$$\varphi^{\pm}(x \to \infty) = 0 \qquad \Rightarrow \qquad \varphi_0^{\pm} = \mp\sqrt{\frac{3}{\kappa}}.$$

Under the latter, specific, boundary conditions at infinity, the solutions (7.4) can be expressed as:

$$\varphi^{\pm}(x) = \pm\sqrt{\frac{3}{\kappa}}\left(\frac{x}{\sqrt{1 + x^2}} - 1\right).$$

These solutions, however, do not meet the necessary symmetry requirements (5.1), (5.2), (5.5). In fact, the scalar field solutions (7.3), (7.4), do not respect invariance under $x \mapsto -x$,

unless the constants $\varphi_0^\pm$ are set to zero: $\varphi_0^\pm = 0$.[4] Hence, for instance, the appropriate boundary condition at infinity for $\varphi^\pm(x)$ in (7.4) will be,

$$\varphi^\pm(x \to \infty) = \varphi_\infty^\pm = 0,$$

instead of the above written one.

Summarizing: particular exact solutions of equation (4.4), for the minimally interacting case ($\xi = 0$), are:

$$\varphi^\pm(x) = \pm\sqrt{\frac{3}{\kappa}}\, E\left(\frac{x}{\sqrt{1+x^2}}, k\right), \quad k \neq 1,$$

$$\varphi^\pm(x) = \pm\sqrt{\frac{3}{\kappa}}\, \frac{x}{\sqrt{1+x^2}}, \qquad k = 1. \tag{7.5}$$

These solutions for the scalar field are odd under (5.1), (5.2), (5.5): $\varphi^\pm(x) = -\varphi^\pm(-x)$. Furthermore, self–interaction potential V is asymptotically constant and negative (see fig. 1). This result is consistent with the fact that our geometry (2.4) is asymptotically AdS_5.
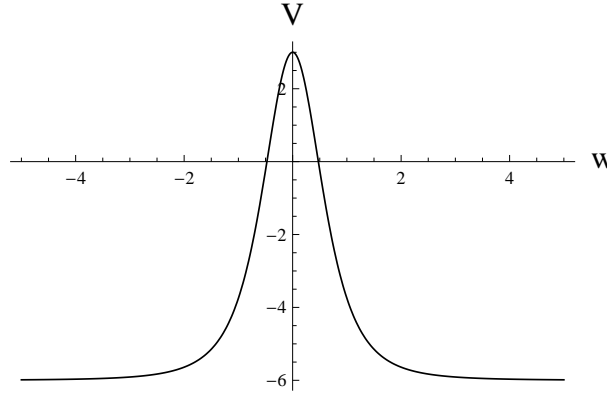

Figure 1. The self–interaction potential for the case $\epsilon \neq 0$ and $\xi = 0$. In this figure we set $k = 1$, $a_0 = 1$, $b = 1$ and $\kappa = 1/2$.

7.2 Non-minimal Coupling Case ($\xi \neq 0$ and $\epsilon = 0$)

If we make the same replacements $x \mapsto \tau = \arctan x$ (Eq. (7.1)), then the master equation (4.4) can be rewritten in the following form:

$$\varphi\ddot{\varphi} + \left(\frac{\xi - \kappa}{\xi}\right)\dot{\varphi}^2 = \frac{3}{2}\varphi^2 - \frac{3}{\xi}, \tag{7.6}$$

Then, without loss of generality, one can make the following assumption:

$$\dot{\varphi}^2 = h(\varphi) \quad \Rightarrow \quad \ddot{\varphi} = \frac{1}{2}\frac{dh(\varphi)}{d\varphi}, \tag{7.7}$$

so that Eq. (7.6) can be written as a first-order differential equation,

$$\frac{dh(\varphi)}{d\varphi} + 2\omega\frac{h(\varphi)}{\varphi} = 3\varphi - \frac{6}{\xi\varphi}, \tag{7.8}$$

---

[4]Recall that, $E(\sin\tau, k) = -E(-\sin\tau, k)$.

where

$$\omega = \frac{\xi - \kappa}{\xi}.$$

The following expression for $h(\varphi)$ solves Eq. (7.8):

$$h(\varphi) = \frac{3\varphi^2}{2(1+\omega)} - \frac{3}{\xi\omega} + C\,\varphi^{-2\omega}, \tag{7.9}$$

where C is an arbitrary constant.

Here, for the sake of simplicity of mathematical handling, we set $C = 0$ in Eq. (7.9). Then, after substituting back (7.7) into (7.9), one is left with the following first order differential equation:

$$\dot{\varphi} = \pm\sqrt{\frac{3}{2(1+\omega)}}\sqrt{\varphi^2 - \frac{2(1+\omega)}{\xi\omega}}. \tag{7.10}$$

Eq. (7.10) can be integrated in quadratures:

$$\pm \int \frac{d\varphi}{\sqrt{\varphi^2 - \frac{2}{\xi}\left(\frac{2\xi-\kappa}{\xi-\kappa}\right)}} = \sqrt{\frac{3\xi}{2(2\xi - \kappa)}}\,\tau + C_1, \tag{7.11}$$

where we have returned to the original parameter ξ, and C_1 is an integration constant, which, in the following calculations we set to zero to meet the imposed symmetry requirements (5.1), (5.2), and (5.5). Hence, depending on the interval in ξ-parameter, one will obtain different particular exact solutions:

1. $0 < \xi \leq \kappa/2$,

$$\varphi^{\pm}(\tau) = \sqrt{\frac{2(\kappa - 2\xi)}{\xi(\kappa - \xi)}}\,\cos\left(\pm\sqrt{\frac{3\xi}{2(\kappa - 2\xi)}}\,\tau\right), \tag{7.12}$$

or, in terms of the original dimensionless variable (see Eq. (7.1)):

$$\varphi^{\pm}(x) = \varphi_{01} \cos\left(\pm\beta_1 \arctan x\right), \tag{7.13}$$

where, for compactness of writing, we have introduced the following constants:

$$\varphi_{01} = \sqrt{\frac{2(\kappa - 2\xi)}{\xi(\kappa - \xi)}}, \qquad \beta_1 = \sqrt{\frac{3\xi}{2(\kappa - 2\xi)}}.$$

The bound (4.2) on φ^2 leads to:

$$\varphi_{01}^2 = \frac{2(\kappa - 2\xi)}{\xi(\kappa - \xi)} \leq \frac{2}{\xi} \qquad \Rightarrow \qquad \kappa - 2\xi \leq \kappa - \xi,$$

which, for $0 < \xi \leq \kappa/2$, is always fulfilled. Hence, positivity of $L(\varphi)$ does not impose any additional constraint on the parameter ξ. On the other hand, for the above profile of the field the self–interaction potential is constant and negative at $w \to \infty$ (see fig. 2).
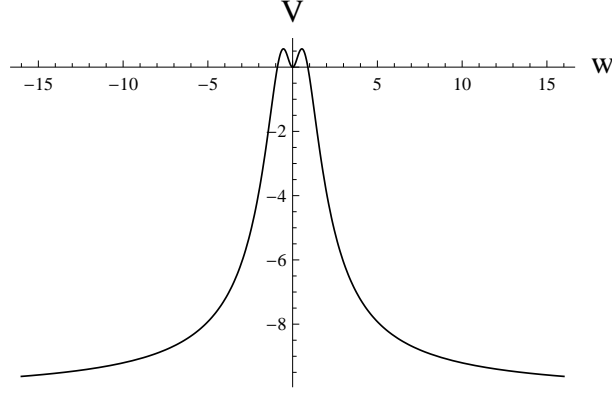
Figure 2. V corresponding to solution (7.13). In this figure we set $\xi = 1/10$, $a_0 = 1$, $b = 1$ and $\kappa = 1/2$.

2. $\xi > \kappa$,

$$\varphi^{\pm}(\tau) = \sqrt{\frac{2(2\xi - \kappa)}{\xi(\xi - \kappa)}} \; \cosh\left(\pm\sqrt{\frac{3\xi}{2(2\xi - \kappa)}} \; \tau\right), \tag{7.14}$$

or, in terms of $x = bw$,

$$\varphi^{\pm}(x) = \varphi_{02} \cosh(\pm\beta_2 \arctan x), \tag{7.15}$$

where,

$$\varphi_{02} = \sqrt{\frac{2(2\xi - \kappa)}{\xi(\xi - \kappa)}}, \qquad \beta_2 = \sqrt{\frac{3\xi}{2(2\xi - \kappa)}}.$$

In this case, positivity of the coupling function $L(\varphi)$, imposes the following non-algebraic constraint on ξ:

$$\cosh^2\left(\frac{\pi}{2}\sqrt{\frac{3\xi}{2(2\xi - \kappa)}}\right) \leq \frac{3\xi}{2\kappa}\left(\frac{\xi - \kappa}{2\xi - \kappa}\right).$$

In this case the self–interaction potential is asymptotically positive (see fig. 3).

In the remaining cases, i. e., when either $\xi = -\bar{\xi} < 0$ ($\bar{\xi} > 0$), or $\kappa/2 < \xi < \kappa$, the resulting scalar field is an imaginary quantity so that, since we are interested only in real scalar field configurations, then the corresponding solutions will not be considered here.

In this case, the scalar field solutions (7.13), and (7.15), both respect the 'mirror' symmetry (5.1)-(5.5), but, unlike in the case 7.1, φ is symmetric under $x \mapsto -x$: $\varphi^{\pm}(x) = \varphi^{\pm}(-x)$.

7.3   General Case: $\xi \neq 0$, $\epsilon \neq 0$

In this case we shall construct solutions that involve both the non–minimal coupling of the scalar field to gravity and the presence of the Gauss-Bonnet term. Despite the fact of the highly non–linear character of the relevant differential equations, we shall be able to obtain exact solutions that describe smooth braneworld configurations for this model.
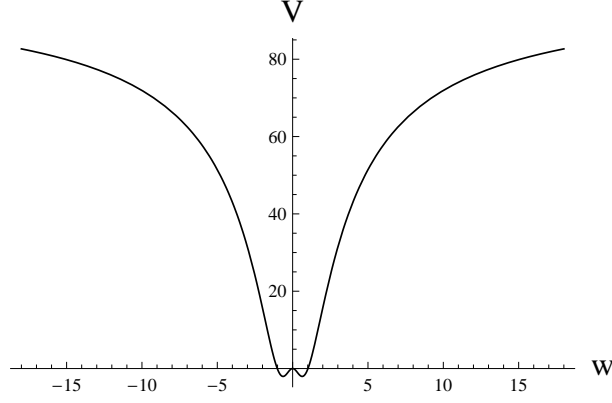
Figure 3. V corresponding to solution (7.15). We set $\xi = 10$, $a_0 = 1$, $b = 1$ and $\kappa = 1/2$.

7.3.1 Particular solution: case $\omega = -1$

If one sets $y = \text{arcsinh } x$ and $\varphi = \psi^{1/(1-\omega)}$, then (4.4) can be easily solved for the particular case when $\omega = -1$ or, equivalently, $\kappa = 2\xi$ since in this case we get a second order linear differential equation. Actually, by changing the variable $x = \sinh y$, hence, $\cosh y = \sqrt{1 + x^2}$, we get the following relevant equation for φ:

$$\varphi \left(\ddot{\varphi} + \tanh y \; \dot{\varphi}\right) - \omega \left(\dot{\varphi}\right)^2 - \frac{3}{2} \, \text{sech}^2 y \; \varphi^2 = \frac{3}{\xi} \, \text{sech}^2 y \; \left(\frac{4\epsilon b^2}{a_0^2} \tanh^2 y - 1\right). \tag{7.16}$$

By further performing the following substitution $\varphi = \psi^{1/(1-\omega)}$ we get

$$\ddot{\psi} + \tanh y \; \dot{\psi} - \frac{3}{2} \left(1 - \omega\right) \text{sech}^2 y \; \psi = \frac{3}{\xi} \left(1 - \omega\right) \text{sech}^2 y \; \left(k^2 \tanh^2 y - 1\right) \; \psi^{\frac{\omega+1}{\omega-1}}, \tag{7.17}$$

where $k^2 = \frac{4\epsilon b^2}{a_0^2}$. By choosing the particular case $\omega = -1$ we actually linearize this differential equation and get

$$\ddot{\psi} + \tanh y \; \dot{\psi} - 3\text{sech}^2 y \; \psi = \frac{6}{\xi} \text{sech}^2 y \; \left(k^2 \tanh^2 y - 1\right). \tag{7.18}$$

This equation has the following real solution:

$$\psi = \varphi^2 = \frac{14 - 10k^2 + 6k^2 \text{sech}^2 y}{7\xi} + C_1 \cosh\left(A\right) + C_2 \sinh\left(A\right), \tag{7.19}$$

where $A = 2\sqrt{3} \arctan\left(\tanh \frac{y}{2}\right)$, and C_1 and C_2 are arbitrary constants. By going back to the variable x we get the solution for the field φ:

$$\varphi = \sqrt{\frac{14 - 10k^2}{7\xi} + \frac{6k^2}{7\xi\left(1 + x^2\right)} + C_1 \cosh\left(A\right) + C_2 \sinh\left(A\right)}, \tag{7.20}$$

where now $A = 2\sqrt{3} \arctan\left(\frac{\sqrt{x^2 + 2 - 2\sqrt{x^2 + 1}}}{x}\right)$.

This solution fulfills the symmetry requirements (5.1)-(5.5) when $C_2 = 0$. Similarly to previous cases, V is asymptotically constant as it is shown in fig. 4.
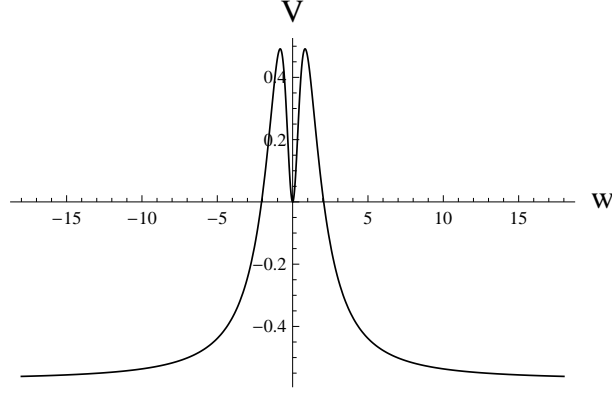
Figure 4. V associated to solution (7.20). We set $C_1 = 0$, $C_2 = 0$, $\xi = 1/4$, $a_0 = 1$, $b = 1$, $\epsilon = 1/3$ and $\kappa = 1/2$.

7.3.2 Solutions for arbitrary ω

Let us recall that if we perform the coordinate transformation $x \mapsto \tau = \arctan x$ (see Eq. (7.1)) for the case in which the Gauss-Bonnet term is non–trivial, then the master equation (4.4) adopts the form:

$$\varphi\ddot{\varphi} + \left(\frac{\xi - \kappa}{\xi}\right)\dot{\varphi}^2 = \frac{3}{2}\varphi^2 - \frac{3}{\xi}\left(1 - k^2\sin^2\tau\right), \tag{7.21}$$

which will be from now on the relevant differential equation to be solved.

Let us now consider more general solutions in which the value of the parameter ω is arbitrary. When dealing with equation (7.21), we can perform the following transformation in order to reduce the order of this differential equation

$$\dot{\varphi}(\tau) = \varphi(\tau)f(\tau), \tag{7.22}$$

where $f(\tau)$ is an arbitrary function of τ. We further can redefine the scalar field φ as follows

$$\varphi^2 = \psi(\tau). \tag{7.23}$$

The relevant equation (7.21) transforms into

$$\frac{1}{2}f\dot{\psi} + \left(\dot{f} + \omega f^2 - \frac{3}{2}\right)\psi = -\frac{3}{\xi}\left(1 - k^2\sin^2\tau\right), \tag{7.24}$$

With the aid of these two transformations, the second order differential equation (7.21) can be reduced to a system of first order differential equations, namely, to a non–linear first orderdifferential equation (general Riccati equation) for $f(\tau)$ and a linear one for the new function $\psi(\tau)$:

$$\dot{f} + \omega f^2 - \frac{3}{2} = \frac{1}{2}fH(\tau), \tag{7.25}$$

$$\dot{\psi} + H(\tau)\psi = -\frac{3\left(1 - k^2\sin^2\tau\right)}{\xi f}, \tag{7.26}$$

where $H(\tau)$ is a function of τ. Solving this system is more easy than solving the original differential equation (7.21) if we make a suitable choice of the function $H(\tau)$.

By setting $H(\tau) = 2c_1/f$, with $c_1 = const.$ in the general Riccati equation (7.25) we get the following solution for f:

$$f = -\sqrt{\frac{l}{\omega}} \tan\left[\sqrt{l\omega}\,(\tau - \tau_0)\right], \tag{7.27}$$

where $2l = -(2c_1 + 3)$ and τ_0 is an arbitrary phase. We further substitute this solution into the linear equation (7.26) for $\psi(\tau)$ and proceed to solve it. The general solution for this equation with arbitrary l possesses a lengthy expression and is given in terms of products of several hypergeometric and exponential functions combined with a sine at certain power. For the sake of simplicity we shall restrict to the case in which $l = 1/\omega$, getting the following solution:

$$\psi = \frac{6\omega\left[4 + 3\omega - (2 + 3\omega)k^2 \sin^2(\tau - \tau_0)\right]}{(4 + \omega)(2 + \omega)} + c_2 \sin^{-(2+3\omega)}(\tau - \tau_0), \tag{7.28}$$

where c_2 is an arbitrary constant. Once we have a solution for ψ we can get back to the original field $\varphi = \sqrt{\psi}$ according to (7.23). This solution will be subject to the relation (7.22) which implies that $c_2 = 0$, leading to:

$$\psi = \frac{6\omega\left[4 + 3\omega - (2 + 3\omega)k^2 \sin^2(\tau - \tau_0)\right]}{(4 + \omega)(2 + \omega)}. \tag{7.29}$$

Moreover, a further relation between k and ω completely determines the solution for φ. For instance, when $k^2 = \frac{4+3\omega}{2+3\omega}$, we get a sine or cosine function depending on the phase constant τ_0. It should be pointed out that both the original differential equation (7.21) and (7.26) are invariant under a suitable simultaneous rescaling of the field φ and the parameter ξ. This fact that can be used to arrive at the following particular solutions for the field φ:

$$a) \quad \varphi = \sqrt{\frac{3}{\xi|\omega|}} \sin(\tau), \qquad k^2 = 1 - \frac{5}{2|\omega|} \qquad \text{with} \quad \omega < -5/2; \tag{7.30}$$

$$b) \quad \varphi = \sqrt{\frac{6}{5\xi}} \cos(\tau), \qquad k^2 = 1 + \frac{2\omega}{5} \qquad \text{with} \quad \omega > -5/2. \tag{7.31}$$

In fig. 5 we show the profile of the V for the solution (7.31)[5]. Again, it is constant and negative at $w \to \infty$. There is another solution of the same type that is valid for the special value $\omega = -11/8$:

$$c) \quad \varphi = B\cos(2\tau) + C, \tag{7.32}$$

where the following constants

$$B_\pm = \pm\sqrt{\frac{3}{154\xi}\left[7\left(2 - k^2\right) \pm 2\sqrt{4k^4 + 49\left(1 - k^2\right)}\right]},$$

$$C_\pm = \pm\frac{\sqrt{33}\,k^2}{\sqrt{14\,\xi\left[7\left(2 - k^2\right) \pm 2\sqrt{4k^4 + 49\left(1 - k^2\right)}\right]}},$$

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[5] For solution (7.30) of the scalar field the qualitative behavior of V is similar.
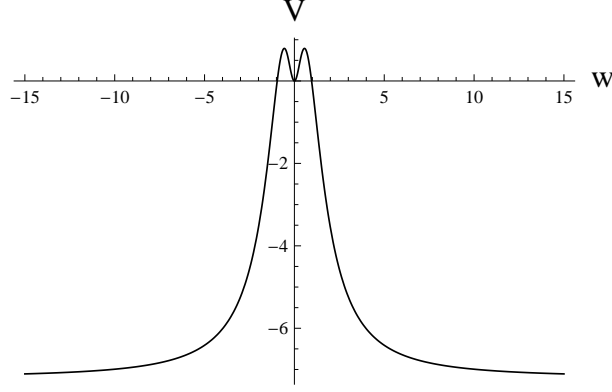
Figure 5. V associated to solution (7.31). We set $\xi = 1/3$, $a_0 = 1$, $b = 1$, $\epsilon = 1/5$ and $\kappa = 1/2$.

must have the same sign (provided that the radicand is positive) since one can show that both parameters ξ and k can be expressed as

$$\xi = \frac{6}{11B^2 + 14BC + 3C^2},$$
$$k^2 = \frac{28BC}{11B^2 + 14BC + 3C^2},$$

implying that the left hand sides of these equalities must be positive.

One can look for more complex solutions than the ones displayed here. However, the relations that define the involved integration constants become more and more lengthy and difficult.

From the above obtained exact solutions, only solutions (7.30) and (7.31) meet the symmetry conditions (5.1)-(5.5).

8 Gravitational Fluctuations

In order to study the localization properties of gravity, the analysis of gravitational fluctuations is needed. In 4D standard cosmology, the fluctuations of the geometry can be classified into scalar, vector and tensor modes with respect to the three–dimensional rotation group [57, 58]. This fact makes more feasible the study of the metric fluctuations because at the linear level the dynamical equations of the scalar, vector and tensor modes are decoupled. In our case, since the 4D geometry is Poincaré invariant, the fluctuations of the metric may be also classified into scalar, vector and tensor modes with respect to the transformations of this symmetry group (see [59] for details).

Let us consider the fluctuations of both the metric and the scalar field around the gravitational background specified in (2.4) and the equations (2.5). In other words, the perturbed geometry has the following form

$$ds_p^2 = \left(a^2(w)\eta_{AB} + H_{AB}\right) dx^A dx^B, \tag{8.1}$$

while the fluctuation of the scalar field is

$$\varphi_p = \varphi + \chi.$$

The functions H_{AB} and χ are the metric and the field fluctuations, respectively. The index p denotes perturbed quantities.

By taking into account the 4D Poincaré symmetry of our background metric (2.4), the fluctuations can be written as

$$H_{AB} = H_{AB}^{(S)} + H_{AB}^{(V)} + H_{AB}^{(T)},$$

where

$$H_{AB}^{(S)} = a^2(w) \begin{pmatrix} 2\left(\eta_{\mu\nu}\psi + \partial_\mu\partial_\nu E\right) & \partial_\mu C \\ \partial_\mu C & 2\zeta \end{pmatrix}, \tag{8.2}$$

$$H_{AB}^{(V)} = a^2(w) \begin{pmatrix} \left(\partial_\mu f_\nu + \partial_\nu f_\mu\right) & D_\mu \\ D_\mu & 0 \end{pmatrix}, \tag{8.3}$$

$$H_{AB}^{(T)} = a^2(w) \begin{pmatrix} 2h_{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix}. \tag{8.4}$$

The upper indices S, V and T denote the scalar, vector and tensor parts of the fluctuations, respectively. The tensor $h_{\mu\nu}$ is transverse and traceless with respect to the 4D Minkowski metric $\eta_{\mu\nu}$, in other words

$$h_\mu^\mu = 0, \qquad \partial_\nu h_\mu^\nu = 0. \tag{8.5}$$

Also, the vectors f_μ and D_μ are divergence free

$$\partial^\mu f_\mu = 0, \qquad \partial^\mu D_\mu = 0. \tag{8.6}$$

The four remaining functions ψ, E, C an ζ are scalars with respect to 4D Poincaré transformations.

The relations (8.2)–(8.6) tell us that, apparently, we only have 10 independent degrees of freedom of the metric fluctuations. Moreover, the covariance of our setup implies that the gravitational perturbation theory has some gauge degrees of freedom [57, 58]. In our case we can completely fix 5 of the above 10 degrees of freedom by making use of this gauge freedom. Let us choose what is known as the longitudinal gauge

$$E = 0, \quad C = 0, \quad f_\mu = 0. \tag{8.7}$$

Thus, we finally have only 5 independent degrees of freedom.

8.1 Tensor modes

As in [60] the equation for the evolution of the tensor modes is

$$\Psi_{\mu\nu}'' - \frac{(\sqrt{s})''}{\sqrt{s}}\Psi_{\mu\nu} - \frac{r}{s}\square^\eta\Psi_{\mu\nu} = 0, \tag{8.8}$$

where $\Psi_{\mu\nu} = \sqrt{s(w)}h_{\mu\nu}$ with $s(w) = a^3 q$. In addition, $r(w) = a^3\left(q + \frac{(q-L)'}{2\mathcal{H}}\right)$ and $\square^\eta$ denotes the d'Alembertian with respect to the metric η.

In order to study the mass spectrum of these fluctuations let us consider the following separation of variables $\Psi_{\mu\nu} = \vartheta(w)\epsilon_{\mu\nu}(x)$. In terms of this new variable the equation (8.8) takes the form

$$\square^\eta\epsilon_{\mu\nu}^m + m^2\epsilon_{\mu\nu}^m = 0, \tag{8.9}$$

$$\vartheta_m'' - \frac{(\sqrt{s})''}{\sqrt{s}}\vartheta_m + m^2\frac{r}{s}\vartheta_m = 0, \tag{8.10}$$

where $\epsilon_{\mu\nu}^m(x)$ is a 4D tensor mode with mass m; on the other hand, $\vartheta_m(w)$ is the 5D profile of the field $\Psi_{\mu\nu}$ and characterizes its localization properties. The equation (8.10) can be interpreted as a Sturm–Liouville eigenvalue problem with the associated norm:

$$\langle\vartheta|\vartheta\rangle = \int_{-\infty}^{\infty} \frac{r}{s}\,\vartheta^2\,dw. \tag{8.11}$$

The zero mode $\epsilon_{\mu\nu}^0(x)$ is the massless 4D graviton. This mode is localized on the brane if $\langle\vartheta_0|\vartheta_0\rangle = \int_{-\infty}^{\infty} \frac{r}{s}\,\vartheta_0^2\,dw$ is finite. It is not difficult to show that $\vartheta_0 = \sqrt{s}$, then, the localization condition for the massless graviton is

$$\langle\vartheta_0|\vartheta_0\rangle = \int_{-\infty}^{\infty} a^3\,q\,dw - 4\,\epsilon\,[a']_{-\infty}^{\infty} + 8\,\epsilon\int_{-\infty}^{\infty} \frac{a'^2}{a}\,dw. \tag{8.12}$$

As one can observe the above expression is quite similar to (3.5). Therefore, for the geometry described in (4.3), we have that if the 4D massless graviton is localized on the brane, then the 4D Planck mass is finite.

Let us investigate the localization properties of the massless graviton in the backgrounds studied in section 7. The situation where $\xi = 0$, $\epsilon \neq 0$ and $k = 1$ was analyzed in [61]. In this work it was shown that the zero mass graviton is localized on the brane. Furthermore, by substituting the expression (7.3) for $k \neq 1$ into (8.12) we obtain the following result

$$\langle\vartheta_0|\vartheta_0\rangle = \frac{2a_0^3}{b}\left(1 + \frac{k^2}{3}\right). \tag{8.13}$$

The above expression tells us that the 4D graviton is localized on the brane.

In the following we shall consider the opposite case where $\xi \neq 0$ and $\epsilon = 0$. The normalization condition depends of the behavior of the integrand $a^3 L(\varphi)$. With regard to the study of gravity localization, the backgrounds defined in (7.13) and (7.15) are similar because both have a regular and finite non–minimal coupling function $L(\varphi)$. Therefore, the convergence of (8.12) is completely determined by the a^3 behavior when $w \to \infty$. For the warp factor (4.3) $a^3 \sim 1/|w|^3$ at infinity. This fact implies again that the massless tensorial mode is localized on the brane.

Finally, let us study the general case ($\xi \neq 0$ and $\epsilon \neq 0$). Some exact solutions for this general situation are shown in (7.20) and (7.30)–(7.32). The norm for the zero mass mode takes the following form

$$\langle\vartheta_0|\vartheta_0\rangle = \int_{-\infty}^{\infty} a^3\,(L-1)\,dw + \langle\vartheta_0|\vartheta_0\rangle_{GB},$$

where $\langle\vartheta_0|\vartheta_0\rangle_{GB}$ is the norm written in (8.13). On the other hand, the function L associated to (7.20) and (7.30)–(7.32) is finite along the extra dimension, thus, like in the previous cases the normalization condition depends on the behavior of a^3 at infinity. From this fact we can conclude that the zero tensor mode is also normalized.

As we mentioned above, in our model the graviton localized on the brane is synonymous of a finite 4D Planck mass. But, what about with the remaining coupling constants of (6.3)? The constants λ_1 and λ_2 are bounded due to the functional dependence of the warp factor with respect to w (4.3). Additionally, for all our solutions, $|\varphi|$, $|\varphi'|$ and $|V|$ are finite along the extra dimension and the self–interaction potential is asymptotically constant, thus, the behavior of the warp factor determines whether or not the effective 4D cosmological constant Λ_4 is well-defined. The warp factor (4.3) has the following form at $w \to \infty$

$$a \sim \frac{1}{|w|}.$$

This fact ensures the convergence of the integral that defines Λ_4 for all the solutions presented in section 7. In other words, the effective cosmological constant is finite.

8.2 Vector modes

In contrast with the tensor sector, the simultaneous presence of the Gauss–Bonnet term and the non–minimal coupling interaction makes much more difficult the study of vector modes. Thus, in this work we will study each case separately and will leave the general case for a future investigation.

8.2.1 Vector modes with the Gauss–Bonnet term only

In this subsection we analyze the case where the non-minimal effects are negligible ($\xi = 0$ and $\epsilon \neq 0$). Using (8.1), (8.3), (8.6) and substituting them in (2.2) we obtain the equations for the vector modes on the longitudinal gauge

$$(D^\nu)' + \left(\frac{q'}{q} + 3\mathcal{H}\right) D^\nu = 0, \tag{8.14}$$

$$\Box^\eta D_\mu = 0, \tag{8.15}$$

where the first relation is a constrain and defines the profile of D_μ along the extra dimension. Furthermore, since there is no mass term in the above second equation, it shows that there is only a massless mode, the graviphoton, with no massive vector fluctuations.

For the case of vector modes, the issue of defining the norm is more indirect than for the tensor sector, therefore it is more appropriate to use the perturbed version of the action (2.1) up to second order with respect to the vector fluctuations [54, 61]. This perturbed action can be written as follows

$$\delta^{(2)} S_V = \int d^4x dw \frac{1}{2} \left(\eta^{\alpha\beta} \partial_\alpha \mathcal{D}^\mu \partial_\beta \mathcal{D}_\mu\right), \tag{8.16}$$

where $\mathcal{D}_\mu = a^{3/2}\sqrt{q} D_\mu$ is called *canonical normal mode*. The zero mode associated to (8.15) is localized on the brane if $\delta^{(2)} S_V$ is finite. By making a suitable variable separation, the zero mass mode takes the form

$$\mathcal{D}^\mu = \frac{v^\mu(x)}{a^{3/2}\sqrt{q}},$$

$$\Box^\eta v^\mu(x) = 0,$$

where $v^\mu(x)$ is the 4D part of the vector sector of fluctuations.

The norm of this mode reads

$$\langle \mathcal{D}^\mu | \mathcal{D}_\mu \rangle = \int_{-\infty}^{\infty} \frac{dw}{a^3 q}. \tag{8.17}$$

When studying the localization properties of the zero mass tensor mode, we learned that it is necessary to have a convergent integral of $a^3 q$. Furthermore, the norm of the massless vector mode has an opposite behavior compared to the tensor mode one. Therefore, if one wishes the 4D graviton to be localized on the brane, the vector sector (represented by its zero mass mode) will necessarily not be localized on it.

8.2.2 Vector modes with the non–minimal coupling only

In this subsection we shall study the situation when $\epsilon = 0$ and $\xi \neq 0$. One way to obtain the mass spectrum of the vector fluctuations consists in passing from the action (2.1) (with $\epsilon = 0$) to the Einstein frame and then, perform the perturbation analysis. In order to do that we apply a conformal transformation as follows

$$\overline{g}_{AB} = L^{2/3} g_{AB}.$$

In the new frame the background action can be written as

$$S \mapsto S_{EF} = \int_{M_5} d^5 x \sqrt{|\overline{g}|} \left\{ -\overline{R} + \tfrac{1}{2}(\overline{\nabla}\sigma)^2 - \overline{V}(\sigma) \right\}, \tag{8.18}$$

where all quantities with an overline are associated to the metric $\overline{g}_{AB}$, σ is the new scalar field and $\overline{V}$ is its self–interaction potential. The relationship between the old and new variables is

$$d\sigma = \sqrt{\frac{1}{L} + \frac{8}{3}\left(\frac{L_\varphi}{L}\right)^2}\, d\varphi, \tag{8.19}$$

$$\overline{V} = \frac{V}{L^{5/3}}.$$

Moreover, the general fluctuations on the Einstein frame are related with the old quantities as follows

$$\overline{H}_{AB} = \frac{2a^2}{3L^{1/3}} \chi L_\varphi \eta_{AB} + L^{2/3} H_{AB}. \tag{8.20}$$

The first term in the above expression does not contribute to the vector sector since it belongs to the scalar sector, then it is easy to obtain that $\overline{D}_\mu = D_\mu$. Thus, the equations for the vector modes are

$$(\overline{D}^\nu)' + 3\overline{\mathcal{H}}\, \overline{D}^\nu = 0,$$
$$\Box^\eta \overline{D}_\mu = 0,$$

where $\overline{\mathcal{H}} = \overline{a}'/\overline{a}$ and $\overline{a} = aL^{1/3}$.

Similarly to the previous case, let us define a new vector variable $\overline{\mathcal{D}}_\mu = \overline{a}^{3/2}\overline{D}_\mu$. Therefore, the norm of the massless mode takes the form

$$\langle \overline{\mathcal{D}}^\mu | \overline{\mathcal{D}}_\mu \rangle = \int_{-\infty}^{\infty} \frac{dw}{\overline{a}^3} = \int_{-\infty}^{\infty} \frac{dw}{a^3 L}.$$

The localization properties of this case are similar to those of the previous one. The localization of the massless tensor mode implies the delocalization of the vector sector of fluctuations.

8.3 Scalar modes

In the longitudinal gauge the scalar fluctuations of the geometry can be expressed as

$$H_{AB}^{(S)} = 2a^2(w) \begin{pmatrix} \eta_{\mu\nu}\psi & 0 \\ 0 & \zeta \end{pmatrix}.$$

Furthermore, the matter provides an extra degree of freedom χ to the scalar sector. Hence, it follows that we have three independent scalar degrees of freedom. One can obtain the

dynamical equations of the scalar fluctuations by perturbing the Einstein and Klein–Gordon equations with respect to the scalar sector. These equations adopt the following form

$$4q\psi'' + 4\psi'\left[q + \frac{q'}{\mathcal{H}} + \frac{\varphi' L_\varphi}{3}\right] + 8\zeta\left[\mathcal{H}'L + \frac{\varphi'^2}{8} - \frac{4\epsilon\mathcal{H}^2}{a^2}(2\mathcal{H}' - \mathcal{H}^2) + \frac{(\varphi' L_\varphi)'}{3}\right]$$
$$+4\zeta'\left[\frac{\varphi' L_\varphi}{3} + \mathcal{H}q\right] + q\Box^\eta\zeta + \frac{8\epsilon}{a^2}(\mathcal{H}' - \mathcal{H}^2)\Box^\eta\psi + \frac{1}{3}\frac{dV}{d\varphi}a^2\chi \tag{8.21}$$
$$+\varphi'\chi' + 4\mathcal{H}'\chi L_\varphi + \frac{1}{3}\left[4(\chi L_\varphi)'' - \Box^\eta(\chi L_\varphi)\right] = 0.$$

$$q\psi'' + \psi'\left[7\mathcal{H}L - \frac{4\epsilon\mathcal{H}}{a^2}\left(2\mathcal{H}' + 5\mathcal{H}^2\right) + \frac{7\varphi' L_\varphi}{3}\right] + \zeta'\left[\frac{\varphi' L_\varphi}{3} + \mathcal{H}q\right]$$
$$+2\zeta\left[(3\mathcal{H}^2 + \mathcal{H}')L - \frac{8\epsilon\mathcal{H}^2}{a^2}(\mathcal{H}' + \mathcal{H}^2) + 2\mathcal{H}\varphi' L_\varphi + \frac{(\varphi' L_\varphi)'}{3}\right] + 2\mathcal{H}(\chi L_\varphi)' \tag{8.22}$$
$$+\frac{1}{3}\frac{dV}{d\varphi}a^2\chi - q\Box^\eta\psi + (\mathcal{H}' + 3\mathcal{H}^2)\chi L_\varphi + \frac{1}{3}\left[(\chi L_\varphi)'' - \Box^\eta(\chi L_\varphi)\right] = 0,$$

$$q\zeta - 2\psi\left[L - \frac{4\epsilon\mathcal{H}'}{a^2}\right] - \chi L_\varphi = 0. \tag{8.23}$$

$$\frac{\varphi'\chi}{2} + 3q(\psi' + \mathcal{H}\zeta) + (\chi L_\varphi)' - \mathcal{H}\chi L_\varphi + L'\zeta = 0. \tag{8.24}$$

$$\chi'' + 3\mathcal{H}\chi' - \Box^\eta\chi + \varphi'[4\psi' + \zeta'] + 2\zeta(\varphi'' + 3\mathcal{H}\varphi')$$
$$-\frac{\partial^2 V}{\partial\varphi^2}a^2\chi - \frac{1}{2}\left\{2(2\mathcal{H}' + 3\mathcal{H}^2)\chi L_{\varphi\varphi} + L_\varphi\left[\Box^\eta\zeta\right.\right. \tag{8.25}$$
$$\left.\left.-3\Box^\eta\psi + 4\zeta(2\mathcal{H}' + 3\mathcal{H}^2) + 4(\psi'' + \mathcal{H}[4\psi' + \zeta'])\right]\right\} = 0.$$

The expressions (8.23) and (8.24) are constraint equations. Then, there is only one independent scalar degree of freedom. As in the vector sector, in this case we will study separately the effects of the Gauss–Bonnet term and the non–minimal coupling on the scalar modes.

8.3.1 Scalar modes with the Gauss–Bonnet term only

Although some aspects of this case were studied in [61], it is helpful to consider some of its details. The master equation of the system can be obtained by using the constraints (8.23), (8.24) and the dynamical equations (8.21), (8.22) and (8.25):

$$\Phi'' - z\left(\frac{1}{z}\right)''\Phi - \left(1 + \frac{q'}{\mathcal{H}q}\right)\Box^\eta\Phi = 0, \tag{8.26}$$

where $\Phi = \frac{a^{3/2}q}{\varphi'}\psi$ and $z = \frac{a^{3/2}\varphi'}{\mathcal{H}}$. In order to study the mass spectrum of Φ let us assume that

$$\Box^\eta\Phi = -m^2\Phi.$$

Thus, the equation (8.26) can be written as

$$\Phi'' - z\left(\frac{1}{z}\right)''\Phi + \left(1 + \frac{q'}{\mathcal{H}q}\right)m^2\Phi = 0. \tag{8.27}$$

The associated norm for the above eigenvalue problem is

$$\langle \Phi | \Phi \rangle = \int_{-\infty}^{\infty} \left(1 + \frac{q'}{\mathcal{H}q} \right) \Phi^2 dw. \tag{8.28}$$

The scalar massless mode is localized on the brane if its norm is finite. In other words

$$\langle \Phi_0 | \Phi_0 \rangle = \int_{-\infty}^{\infty} \left(1 + \frac{q'}{\mathcal{H}q} \right) \Phi_0^2 dw < \infty, \tag{8.29}$$

where $\Phi_0 = \frac{1}{z}$. Let us apply this general analysis to the background solution (7.3). It is not hard to show that when $w \to \infty$

$$\left(1 + \frac{q'}{\mathcal{H}q} \right) \Phi_0^2 \sim \begin{cases} |w|^7, & \text{when} \quad k = 1 \\[2mm] |w|^5, & \text{when} \quad k \neq 1 \end{cases} \tag{8.30}$$

Therefore, in both cases the massless scalar mode is not localized on the brane.

8.3.2 Scalar modes with the non–minimal coupling only

Similarly to the vector modes case, the analysis of the scalar sector without the Gauss–Bonnet term is more easy in the Einstein frame. In this case the metric fluctuations can be expressed as follows

$$\overline{H}_{AB}^{(S)} = 2\overline{a}^2 \begin{pmatrix} \eta_{\mu\nu} \overline{\psi} & 0 \\ 0 & \overline{\zeta}, \end{pmatrix}, \tag{8.31}$$

where $\overline{\psi} = \psi + \frac{\chi L_\varphi}{3L}$ and $\overline{\zeta} = \zeta - \frac{\chi L_\varphi}{3L}$. On the other hand, the scalar field fluctuation in the Einstein frame $\overline{\chi}$ is related to χ as follows

$$\overline{\chi} = \sqrt{\frac{1}{L} + \frac{8}{3} \left(\frac{L_\varphi}{L} \right)^2} \chi.$$

By considering the above expression and the equations (8.21)–(8.25) when $\epsilon = 0$, we obtain the master equation for the scalar sector of fluctuations

$$\overline{\Phi}'' - \overline{z} \left(\frac{1}{\overline{z}} \right)'' \overline{\Phi} + m^2 \overline{\Phi} = 0, \tag{8.32}$$

where $\overline{\Phi} = \frac{\overline{a}^{3/2}}{\sigma'} \overline{\Psi}$ and $\overline{z} = \frac{\overline{a}^{3/2} \sigma'}{\overline{\mathcal{H}}}$. Thus, the norm of zero mode reads

$$\langle \overline{\Phi}_0 | \overline{\Phi}_0 \rangle = \int_{-\infty}^{\infty} \overline{\Phi}_0^2 dw = \int_{-\infty}^{\infty} \frac{1}{\overline{z}^2} dw. \tag{8.33}$$

The localization properties of the solutions (7.13) and (7.15) are similar in the sense that their massless mode has the same behavior at infinity, i.e.

$$\frac{1}{\overline{z}^2} \sim |w|^5, \qquad \text{when} \qquad w \to \infty.$$

The above result tells us that the zero mass scalar mode is not localized on the brane.

9   Conclusions

We present a scenario with a thick braneworld model builded by a scalar field non–minimally coupled to the Einstein-Hilbert term. Furthermore, there is a Gauss-Bonnet term in the bulk. We obtain the whole 4D effective theory coming from the dimensional reduction of the set–up (2.1). We make emphasis in the analysis of relevant quantities such as, 4D Planck mass and the effective cosmological constant. As consequence of the non–minimal coupling between the scalar matter and gravity $M_{\rm Pl}$ and Λ_4 depend explicitly on the profile of the bulk scalar field φ. Therefore, in order for us to be able to rigorously interpret our physical results we need to construct exact solutions for our braneworld configuration. Despite the highly non–linear nature of the relevant differential master equation for the scalar field, we were able to obtain several exact particular solutions, contrary to previous studies that implement approximate solutions and/or particular cases that consider just one of the two effects. In order to elucidate the physical effects of the inclusion of the non–minimal coupling and of the Gauss–Bonnet term, we considered three cases in which we switched on/off the corresponding parameters. For all the constructed solutions, the effective Planck mass and the 4D cosmological constant are finite. We also studied the whole set of gravitational fluctuations – classified into scalar, vector and tensor modes with respect to the 4D Poincaré symmetry group – for our braneworld model. For all the considered backgrounds the massless zero mode of the tensor fluctuations is localized on the brane. This fact allows us to recover 4D gravity in our world. In contrast to this, the massless scalar and vector modes are delocalized and hence, cannot be observed on the brane.

As a further step it will be interesting to study the stability of the field configurations within the the framework of the present braneworld model when both non–linear effects are switched on. Another interesting topic is to consider a non–minimal coupling of the bulk scalar field not only with the Einstein-Hilbert piece of the Lagrangian but, also, with the Gauss-Bonnet term. Investigation of these issues is left for future work.

10   Acknowledgements

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